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The *d*-dimensional Landsberg gas

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Abstract. Expressions based on a recently proposed approximate density of states function are derived for thermodynamic functions of an ideal relativistic Bose gas in d dimensions. The results obtained are compared with those found using the exact density of states and agreement is found to be good in the three-dimensional case but less good in higher dimensions. It is noted that, in the extreme relativistic and non-relativistic limits, results agree exactly and, in the vicinity of the state of zero kinetic energy, the two densities of states exhibit similar behaviour. Mention is made also of the application of this approximate density of states function to a consideration of the ideal relativistic Fermi gas.

1. Introduction

Recently, interest has revived in the study of the ideal relativistic Bose gas, with particular attention being paid to the phenomenon of Bose condensation (Beckmann *et al* 1979, Aragão de Carvalho and Goulart Rosa 1980a, b). Much of this renewed interest stems from a discussion of the cosmological implications of a massive primordial photon gas by Kuzmin and Shaposhnikov (1979) and from problems associated with quarks and quark confinement. (For a review of statistical mechanics at high energy density see Sertorio (1979).)

Subsequently, a review of the ideal relativistic Bose condensation, together with a discussion of some related matters, has been presented by Landsberg (1980). In this review, the normal theory is generalised to arbitrary discrete spectra before specialising to a specific density of states function. A discussion of the ideal relativistic Bose gas in d dimensions follows, and it is shown that the condensation phenomenon becomes more pronounced both as the extreme relativistic limit is approached and when higher dimensions are considered. It is found that, for particles of small rest mass m_0, there is no condensation for $d=2$ unless $m_0=0$. Also, no discontinuity is found in the constant volume heat capacity at the condensation temperature when $d=3$ or 4, again unless $m_0=0$. It is in this review also that the approximate density of states function, to be discussed here, is proposed.

In the articles by Beckmann *et al* and by Aragão de Carvalho and Goulart Rosa, evaluation of the integrals involved in any discussion of ideal relativistic Bose condensation is achieved through recourse to an expansion of the distribution function, with use of the integral representation for the modified Bessel functions of the second kind. The important point to emerge from these discussions is that there is a qualitative difference between massive and massless Bose gases at the condensation temperature, a point made already in Landsberg and Dunning-Davies (1965a). It should be noted that the extreme relativistic approximate expression for the mean total number of particles given in this latter paper (correctly criticised by Aragão de Carvalho and Goulart Rosa (1980b)) does not affect any other part of that paper, since the exact result was used in plotting the graph of the condensation temperature against rest mass for various concentrations.

There are several exact treatments of the relativistic gases available (Bauer *et al* 1974, Nieto 1969, Elze *et al* 1980) and, for the non-degenerate case, an approximate formulation was suggested and discussed by Hönl (1971). In the present paper, the thermodynamic functions for an ideal relativistic Bose gas in d dimensions are derived in terms of the approximate density of states alluded to above. The results are compared with those obtained using the exact density of states, and the approximation is found to be surprisingly good, particularly in the three-dimensional case. Also, it is noted that in the vicinity of the state of zero kinetic energy, the approximate and exact densities of state exhibit similar behaviour. In the final section, mention is made of the application of this approximate density of states function to a consideration of the ideal relativistic Fermi gas.

2. The Landsberg (Bose) gas

Expressions for the thermodynamic properties of the ideal relativistic quantum gases are well known. However, although much information is obtainable from them, the integrals appearing in these expressions do not admit analytic solution. As an alternative to the approximations usually considered, it is proposed to evaluate exactly the properties of an ideal relativistic Bose gas in d dimensions based on the approximate density of states (see Landsberg 1980),

$$D(\eta, u, T)\,\mathrm{d}\eta = A_d[\eta^{d-1} + \tfrac{1}{2}(2u)^{d/2}\eta^{d/2-1}]\,\mathrm{d}\eta \qquad (2.1)$$

where

$$A_d \equiv \frac{2\pi^{d/2}\omega V_d}{\Gamma(d/2)}\left(\frac{kT}{hc}\right)^d,$$

V_d being the d-dimensional volume, $\eta kT \equiv e$, $ukT \equiv e_0 = mc^2$, the rest energy, and ω the degeneracy factor.

This system will be called the Landsberg gas. As is immediately obvious, this approximate density of states interpolates between the extreme relativistic and non-relativistic cases and has the correct limiting behaviour as the rest energy, e_0, tends to zero and to infinity.

The thermodynamic properties of a Bose gas of interest in the present investigation are given in table 1 for both exact and approximate densities of states, the integrals $I(\alpha, s, \pm)$ and $\theta(\alpha, s, \pm)$ being defined by

$$I(\alpha, s, \pm) = \frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s\,\mathrm{d}x}{\mathrm{e}^{x-\alpha} \pm 1}$$

and

$$\theta(\alpha, s, \pm) = \Gamma(d+s-1)I(\alpha, d+s-3, \pm) + \tfrac{1}{2}(2u)^{d/2}\Gamma(\tfrac{1}{2}d+s-1)I(\alpha, \tfrac{1}{2}d+s-3, \pm).$$

In the special case of three dimensions, equation (2.3*b*) may be rearranged to give

$$T_c n^{-1/3} = \frac{hc}{k}\{4\pi[2\zeta(3) + \tfrac{1}{4}(2u_c)^{3/2}\pi^{1/2}\zeta(\tfrac{3}{2})]\}^{-1/3}. \qquad (2.6)$$

Table 1. Thermodynamic properties of an ideal relativistic Bose gas.

Property	Exact results		The Landsberg gas	
	$N_1(\eta_1, \alpha) + A_d \int_0^\infty \frac{(\eta+u)(\eta^2+2\eta u)^{d/2-1}}{e^{\eta-\alpha}-1} d\eta$	(2.2*a*)	$N_1(\eta_1, \alpha) + A_d[\Gamma(d) I(\alpha, d-1, -) + \frac{1}{2}(2u)^{d/2}\Gamma(\frac{1}{2}d) I(\alpha, \frac{1}{2}d-1, -)]$	(2.2*b*)
N	↓ for $T = T_c$		↓ for $T = T_c$	
	$A_d\left(\frac{T_c}{T}\right)^d \int_0^\infty \frac{(\eta+u_c)(\eta^2+2\eta u_c)^{d/2-1}}{e^{\eta}-1} d\eta$	(2.3*a*)	$A_d\left(\frac{T_c}{T}\right)^d \left[\Gamma(d)\zeta(d) + \frac{1}{2}(2u_c)^{d/2}\Gamma\left(\frac{d}{2}\right)\zeta\left(\frac{d}{2}\right)\right]$	(2.3*b*)
U	$A_d kT \int_0^\infty \frac{(\eta^2+\eta u)(\eta^2+2\eta u)^{d/2-1}}{e^{\eta-\alpha}-1} d\eta$	(2.4*a*)	$A_d kT\left[\Gamma(d+1) I(\alpha, d, -) + \frac{1}{2}(2u)^{d/2}\Gamma\left(\frac{d}{2}+1\right) I\left(\alpha, \frac{d}{2}, -\right)\right]$	(2.4*b*)
Δ	$kA_d\left(\frac{T_c}{T}\right)^d \frac{\left(\int_0^\infty \frac{(\eta^2+\eta u)(\eta^2+2\eta u)^{d/2-1} e^{\eta}}{(e^{\eta}-d\eta)^2} d\eta\right)^2}{\int_0^\infty \frac{(\eta+u)(\eta^2+2\eta u)^{d/2-1} e^{\eta}}{(e^{\eta}-1)^2} d\eta}$	(2.5*a*)	$kA_d\left(\frac{T_c}{T}\right)^d \frac{[\theta(0, 2, -)]^2}{\theta(0, 1, -)}$	(2.5*b*)

Exact results: Extreme relativistic limit	Exact results: Non-relativistic limit	The Landsberg gas: Extreme relativistic limit	The Landsberg gas: Non-relativistic limit
$Nkd^2 \frac{\zeta(d)}{\zeta(d-1)}$	$Nk\left(\frac{d}{2}\right)^2 \frac{\zeta(\frac{1}{2}d)}{\zeta(\frac{1}{2}d-1)}$	$Nkd^2 \frac{\zeta(d)}{\zeta(d-1)}$	$Nk\left(\frac{d}{2}\right)^2 \frac{\zeta(\frac{1}{2}d)}{\zeta(\frac{1}{2}d-1)}$

Hence, T_c may be found for various values of u_c and the concentration $n = N/V$. This was the procedure adopted in a previous article (Landsberg and Dunning-Davies 1965a) where the exact expression shown in (2.3a) was evaluated numerically for the case $d = 3$. These numerically evaluated results, together with the corresponding ones obtained from (2.6), are shown in table 2. They are seen to be in good agreement, the maximum deviation being 9% near $u_c = 1$.

Table 2. Corresponding values of the critical temperature, T_c, obtained via the approximate density of states and numerically via the exact density of states (n is the concentration).

u_c	$T_c n^{-1/3}$(approximate)	$T_c n^{-1/3}$(exact)
10^5	0.00132	0.00132
10^4	0.00417	0.00417
10^3	0.0132	0.0132
10^2	0.0417	0.0416
10	0.131	0.128
5	0.183	0.175
1	0.347	0.318
0.5	0.406	0.374
0.2	0.449	0.432
0.1	0.456	0.441
0.05	0.460	0.452
0.01	0.462	0.460
0.001	0.462	0.462

From (2.4b) and (2.3b), the expression for U/N, evaluated at the critical temperature T_c, is seen to be

$$\frac{U}{N} = \frac{B[\Gamma(d+1)\zeta(d+1) + \frac{1}{2}(2u_c)^{d/2}\Gamma(\frac{1}{2}d+1)\zeta(\frac{1}{2}d+1)]}{[\Gamma(d)\zeta(d) + \frac{1}{2}(2u_c)^{d/2}\Gamma(\frac{1}{2}d)\zeta(\frac{1}{2}d)]^{1+1/d}} \tag{2.7}$$

where $B = [N\Gamma(d/2)h^d c^d / 2\pi^{d/2} V_d]^{1/d}$.

This factor B is seen to appear in the expression for U/N, evaluated at T_c, which is obtained from (2.4a) and (2.3a). Hence, in figure 1, U/NB is plotted against u_c for various values of the number of dimensions d for both exact and approximate densities of states. As is seen, there is good agreement in the case of three dimensions, but this agreement becomes progressively worse as the number of dimensions is increased.

The discontinuity in the constant volume heat capacity at $T = T_c$ is given by

$$\Delta \equiv (C_v)_{T_c^-} - (C_v)_{T_c^+}.$$

Expressions for this in terms of both exact and approximate densities of states are given by (2.5a) and (2.5b) respectively. These equations, together with (2.3a) and (2.3b), have been used to plot the function Δ/NK against the number of dimensions d for various values of u_c and for both exact and approximate densities of states. The resulting curves are shown in figure 2. Obviously, both expressions lead to the same curves in the extreme relativistic ($u_c = 0$) and non-relativistic ($u_c = \infty$) cases. In the intervening region, results obtained by the two routes are seen to agree well except when u_c lies between the values 1 and 10. It might be noted also that, due to the form of the approximate density of states (2.1), the expression for any thermodynamic function

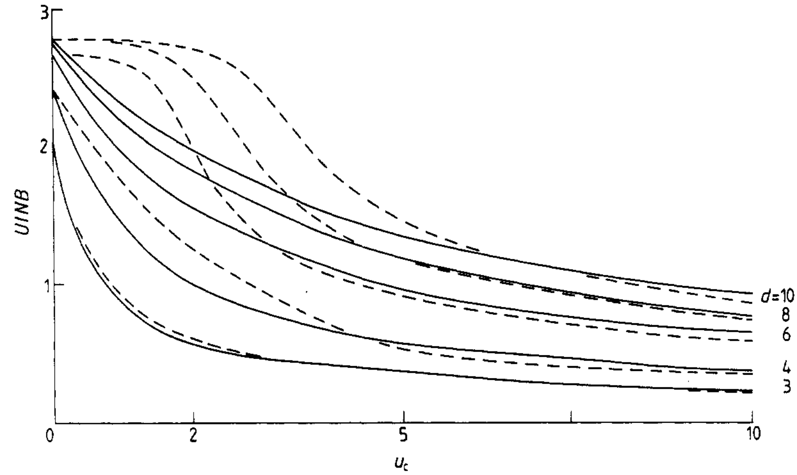

Figure 1. The ratio U/NB (as defined in (2.7)) evaluated at the condensation temperature T_c is plotted against u_c ($= e_0/kT_c$) for various numbers of dimensions d and for both exact (full curves) and approximate (broken curves) densities of states.

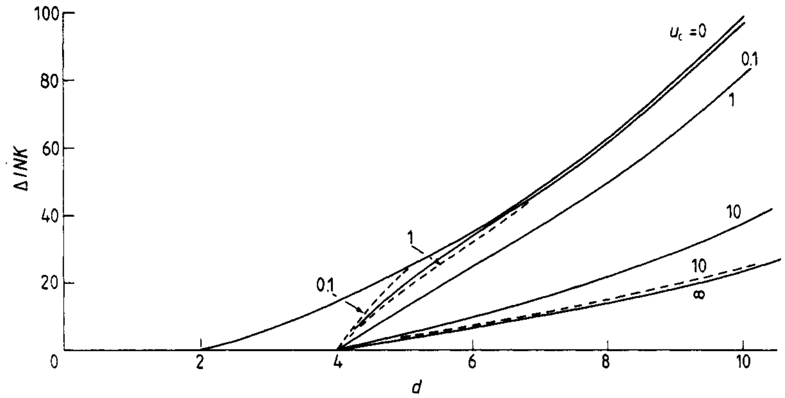

Figure 2. The discontinuity in the constant volume heat capacity per particle at the condensation temperature T_c is plotted against the number of dimensions d for various values of u_c ($= e_0/kT_c$) and for both exact (full curves) and approximate (broken curves) densities of states.

evaluated in either the extreme relativistic or non-relativistic limits will agree with the results derived using the exact density of states.

Again, by examining (2.3*b*) and (2.5*b*), it is seen that

(i) for $d = 1$, there is no condensation;

(ii) for $d = 2$, there is condensation only for $m_0 = 0$ but $\Delta = 0$;

(iii) for $d = 3$, 4, there is condensation for $m_0 = 0$ with $\Delta > 0$, and for $m_0 > 0$ with $\Delta = 0$;

(iv) for $d \geqslant 5$, there is condensation for $m_0 \geqslant 0$ with $\Delta > 0$.

This is in complete agreement with conclusions drawn using the exact results (2.3*a*) and (2.5*a*) (Landsberg 1980).

Recently it has been pointed out (Aragão de Carvalho and Goulart Rosa 1980b) that many of the problems in earlier work may be attributed to approximations being made to the density of states. As far as Bose condensation is concerned, they claim that it, together with the low-temperature behaviour of the system, are determined by the density of states in the neighbourhood of the state of zero kinetic energy, $\eta = 0$. The approximate density of states discussed here behaves in a similar manner to the exact density of states in the neighbourhood of $\eta = 0$. As is pointed out by Aragão de Carvalho and Goulart Rosa (1980b), the exact density of states is a concave function below $\eta = 0.2247u$, being zero and having an infinite derivative at $\eta = 0$. The approximate density of states is a concave function below $\eta = 0.315u$, being zero and having an infinite derivative at $\eta = 0$.

Hence, when considering an ideal relativistic Bose gas, it is found that results obtained via use of the approximate density of states (2.1), originally proposed by Landsberg (1980), are in excellent agreement in the physically realistic case of a three-dimensional gas. However, as the number of dimensions is increased, the agreement becomes progressively worse. Also, it should be noted that what makes this approximation reasonable is the fact that it preserves the essential qualitative features of the system. It simplifies the extreme relativistic and non-relativistic limits and, although not good in the intermediate region, except in the three-dimensional case, it does not alter the main characteristics of the condensation phenomenon. It would seem, therefore, that this approximate density of states could prove useful in determining the behaviour of thermodynamic functions associated with an ideal relativistic Bose gas when numerical evaluation of the exact expressions is not deemed worthwhile. This approximate formulation would have an advantage over the approximation suggested by Hönl (1971) in that it would not be restricted to the non-degenerate case. Recently, Landsberg and Park (1975) used the Hönl equation of state for a relativistic quantum gas in their discussion of an oscillating universe. The present approximation could have been used equally well and might be employed in future examinations of this problem.

The Landsberg gas might prove useful also in investigations of the properties of hadronic matter at high densities. This is due to the fact that the mass spectrum of hadrons, $\rho(m)\,dm$, is taken to have the asymptotic form $cm^a\, e^{bm}\, dm$ (Hagedorn 1965) and, in order to derive expressions for the various thermodynamic functions, it is necessary to integrate expressions such as (2.2a) for the mean total number of particles and (2.4a) for the internal energy over m. The use of the approximate density of states in calculations such as these would obviate the need to introduce modified Bessel functions of the second kind and the subsequent use of the asymptotic expansion of such functions.

3. The Landsberg (Fermi) gas

While the background to the introduction of the approximate density of states function (2.1) is concerned solely with the ideal relativistic Bose gas, (2.1) may be used also in a consideration of the properties of an ideal relativistic Fermi gas.

For such a system, the mean number of particles is given by

$$N = A_d[\Gamma(d)I(\alpha, d-1, +) + \tfrac{1}{2}(2u)^{d/2}\Gamma(\tfrac{1}{2}d)I(\alpha, \tfrac{1}{2}d-1, +)] \tag{3.1}$$

and the internal energy, excluding the rest energy, is

$$U = A_d kT[\Gamma(d+1)I(\alpha, d, +) + \tfrac{1}{2}(2u)^{d/2}\Gamma(\tfrac{1}{2}d+1)I(\alpha, \tfrac{1}{2}d, +)].$$

It follows that the constant volume heat capacity is given by

$$C_v = A_d k\left(\theta(\alpha, 3, +) - \frac{[\theta(\alpha, 2, +)]^2}{\theta(\alpha, 1, +)}\right). \tag{3.2}$$

If the Sommerfeld lemma is applied to the integrals appearing in (3.2), it is found that for a degenerate relativistic Fermi gas in three dimensions

$$C_v = C_{v,\mathrm{nr}} + C_{v,\mathrm{er}} \tag{3.3a}$$

where $C_{v,\mathrm{nr}}$ and $C_{v,\mathrm{er}}$ are the constant volume heat capacities of a degenerate non-relativistic Fermi gas and a degenerate extreme relativistic Fermi gas respectively.

The corresponding result derived using the exact density of states is

$$C_v = [(C_{v,\mathrm{nr}})^2 + (C_{v,\mathrm{er}})^2]^{1/2} \tag{3.3b}$$

(Landsberg and Dunning-Davies 1965b).

It might be noted also that for a degenerate electron gas, formulae (3.3a) and (3.3b) lead to

$$C_v/Nk = 1.6\times 10^{-21}T$$

where a particle density of 10^{24} has been assumed and (3.1) has been used.

For a non-degenerate Fermi gas, (3.2) and (3.1) lead to

$$\frac{C_v}{Nk} = \begin{cases} \frac{3}{2} + 15(2\pi)^{-1/2}(kT/mc^2)^{3/2}\ldots, & \text{non-relativistic case,} \\ 3 + \frac{3}{16}(2\pi)^{1/2}(mc^2/kT)^{3/2}\ldots, & \text{extreme relativistic case.} \end{cases}$$

If the exact density of states is used, the corresponding expressions are

$$\frac{C_v}{Nk} = \begin{cases} \frac{3}{2} + \frac{15}{4}(kT/mc^2)\ldots, & \text{non-relativistic case,} \\ 3 - \frac{1}{2}(mc^2/kT)^2\ldots, & \text{extreme relativistic case,} \end{cases}$$

(Landsberg and Dunning-Davies 1965b). As is immediately obvious, the second terms obtained using the approximate density of states agree with those obtained via the exact density of states neither as far as the coefficients nor the dependence on (mc^2/kT) are concerned. Considering the form of the approximate density of states (2.1), this is not a surprising result. However, using (2.1) as the starting point for a discussion of the properties of an ideal relativistic Fermi gas has the merit that all the integrals appearing are of the form

$$I(\alpha, s, +) = \frac{1}{\Gamma(s+1)}\int_0^\infty \frac{x^s\,\mathrm{d}x}{\mathrm{e}^{x-\alpha}+1}$$

and these are well tabulated (Dingle 1958).

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